

Solar system tests and interpretation of gauge field and Newtonian prepotential in general covariant Hořava-Lifshitz gravity

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In this paper, we first study spherically symmetric, stationary vacuum configurations in general covariant theory ($U(1)$ extension) of Hořava-Lifshitz gravity with the projectability condition and an arbitrary value of the coupling constant λ . We obtain all the solutions with the assumed symmetry in closed forms. If the gauge field A and the Newtonian prepotential φ do not directly couple to matter fields, the theory is inconsistent with solar system tests for $\lambda \neq 1$, no matter how small $|\lambda - 1|$ is. This is shown to be true also with the most general ansatz of spherically symmetric (but not necessarily stationary) configurations. Therefore, to be consistent with observations, one needs either to find a mechanism to restrict λ precisely to its relativistic value $\lambda_{GR} = 1$, or to consider A and/or φ as parts of the 4-dimensional metric on which matter fields propagate. In the latter, requiring that the line element be invariant not only under the foliation-preserving diffeomorphism but also under the local $U(1)$ transformations, we propose the replacements, $N \rightarrow N - v(A - \mathcal{A})/c^2$ and $N^i \rightarrow N^i + N\nabla^i\varphi$, where v is a dimensionless coupling constant to be constrained by observations, N and N^i are, respectively, the lapse function and shift vector, and $\mathcal{A} \equiv -\dot{\varphi} + N^i\nabla_i\varphi + N(\nabla_i\varphi)^2/2$. With this prescription, we show explicitly that the aforementioned solutions are consistent with solar system tests for both $\lambda = 1$ and $\lambda \neq 1$, provided that $|v - 1| < 10^{-5}$. From this result, the physical and geometrical interpretations of the fields A and φ become clear. However, it still remains to be understood how to obtain such a prescription from the action principle.

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I. INTRODUCTION

Einstein's classical general theory of relativity (GR) is consistent with all the experiments and observations carried out so far [1]. However, it has been known for a long time that GR is not (perturbatively) renormalizable [2], and thus can be considered only as a low energy effective theory. Because of the universal coupling of gravity to all forms of energy, it is expected that gravity too should have a quantum mechanical description. Motivated by this strong anticipation, quantization of gravitational fields has been one of the main driving forces in theoretical physics in the past decades in a wide range of approaches [3].

Recently, Hořava [4] proposed a new theory of quantum gravity in the framework of quantum field theory. One of the essential ingredients of the theory is the inclusion of higher-dimensional spatial (but not time) derivative

operators, so that the ultraviolet (UV) behavior is dominated by them and that they render the theory power-counting renormalizable. In the infrared (IR) the lower dimensional operators take over, presumably providing a healthy low energy limit. The exclusion of higher time derivative terms prevents ghost instability [5], but breaks Lorentz symmetry, on the other hand. While the breaking of Lorentz symmetry in the matter sector is highly restricted by experiments/observations, in the gravitational sector the restrictions are much weaker [6] (See also [7]). The Lorentz breaking and hence the power-counting renormalizability are realized by invoking the anisotropic scaling between time and space,

$$t \rightarrow b^{-z}t, \quad \vec{x} \rightarrow b^{-1}\vec{x}. \quad (1.1)$$

This is a reminiscent of Lifshitz scalars [8] in condensed matter physics, hence the theory is often referred to as the Hořava-Lifshitz (HL) gravity. For the theory to be power-counting renormalizable, the critical exponent z has to be $z \geq 3$ [4, 9]. Clearly, such a scaling breaks explicitly the Lorentz symmetry and thus 4-dimensional diffeomorphism invariance. Hořava assumed that it is

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broken only down to, the invariance under

$$t \rightarrow t'(t), \quad \vec{x} \rightarrow \vec{x}'(t, \vec{x}), \quad (1.2)$$

the so-called foliation-preserving diffeomorphism, denoted often by $\text{Diff}(M, \mathcal{F})$. The basic quantities are the lapse function N , the shift vector N^i , and the 3-dimensional spatial metric g_{ij} , as introduced more than 50 years ago by Arnowitt, Deser and Misner [10], in order to quantize gravity.

Once the general covariance is broken, it immediately results in a proliferation of independent coupling constants [4, 11–13], which could potentially limit the predictive power of the theory. To reduce the number of independent coupling constants, Hořava introduced two independent conditions, the *projectability* and the *detailed balance* [4]. The former requires that the lapse function N be a function of t only,

$$N = N(t), \quad (1.3)$$

while the latter requires that the gravitational potential should be obtained from a superpotential W_g , where W_g is given by an integral of the gravitational Chern-Simons term over a 3-dimensional space, $W_g \sim \int_{\Sigma} \omega_3(\Gamma)$. With these two conditions, the general action contains only five independent coupling constants. The detailed balance condition has several remarkable features [4, 13, 14]. For example, it is in the same spirit of the AdS/CFT correspondence [15], where a string theory including gravity defined on one space is equivalent to a quantum field theory without gravity defined on the conformal boundary of this space, which has one or more lower dimension(s). Yet, in the non-equilibrium thermodynamics, the counterpart of the superpotential W_g plays the role of entropy, while $\delta W_g / \delta g_{ij}$ represents the corresponding entropic force [16]. This might shed further lights on the nature of gravitational forces, as proposed recently by Verlinde [17]. For details, we refer readers to Hořava's original paper [4], as well as his review article [18].

When applying the theory to cosmology, various remarkable features were found. (See [19] for a review.) In particular, the higher-order spatial curvature terms can give rise to a bouncing universe [20], may ameliorate the flatness problem [21] and lead to caustic avoidance [22]; the anisotropic scaling provides a solution to the horizon problem and generation of scale-invariant perturbations without inflation [23], a new mechanism for generation of primordial magnetic seed field [24], and also a modification of the spectrum of gravitational wave background via a peculiar scaling of radiation energy density [25]; with the projectability condition, the lack of a local Hamiltonian constraint leads to “dark matter as an integration constant” [26]; the dark sector can also have its purely geometric origins [27]; in the parity-violating version of the theory, circularly polarized gravitational waves can also be generated in the early universe [28]; and so on.

Despite of all the above remarkable features, it was found that the projectability condition leads to several

undesirable properties, including infrared instability [4, 29] and strong coupling [30, 31]¹. All these properties are closely related to the existence of a spin-0 graviton [19, 34].

It should be noted, however, that the infrared instability does not show up under a certain condition [19] and that the strong coupling is not necessarily a problem if nonlinear effects help recovering GR at low energy. Of course, the strong coupling implies that the naive perturbative expansion breaks down and that a proper non-perturbative treatment is needed. In general, non-perturbative analysis is not easy to perform in practice. Nonetheless, in some simplified situations, fully nonlinear analyses were already performed, showing that the $\lambda \rightarrow 1$ limit of the theory is continuous and that GR is recovered in a non-perturbative fashion. Such examples include spherically symmetric, stationary, vacuum configurations [19], a class of exact cosmological solutions [31] and nonlinear superhorizon perturbations [35, 36]. The non-perturbative recovery of GR, explicitly shown in those examples, may be considered as an analogue of the Vainshtein effect [37].

Although the existence of spin-0 graviton after all may not be a problem due to the analogue of the Vainshtein effect, it is interesting and certainly important to seek another possible way out. Motivated by this, Hořava and Melby-Thompson (HMT) [38] extended the symmetry (1.2) to include a local $U(1)$,

$$U(1) \ltimes \text{Diff}(M, \mathcal{F}). \quad (1.4)$$

With this enlarged symmetry, the spin-0 graviton is eliminated [38, 39], and the theory has the same number of propagating degrees of freedom as GR. This was initially done in the special case with $\lambda = 1$, and was soon generalized to the case with any λ [40]. Even with $\lambda \neq 1$, the spin-0 graviton is still eliminated [40, 41]. When applying it to cosmology, various interesting results were found [42]. In particular, the Friedmann-Robertson-Walker (FRW) universe is necessarily flat in such a setup, provided that the coupling of the $U(1)$ field to a scalar matter field is described by the recipe given in [40].

In this paper, we shall consider two important issues in the general covariant theory of the HL gravity with the projectability condition (1.3) and an arbitrary coupling constant λ [38, 40, 41]: (i) the solar system tests; and (ii)

¹ Note that even without the projectability condition the theory is still strongly coupled [12, 32], although instability can be avoided by inclusion of the term $a_i a^i$ [11], where $a_i = N_{,i}/N$. On the other hand, as mentioned above, abandoning the projectability condition results in a proliferation of independent coupling constants. To render this problem, Zhu, Wu, Wang, and Shu recently introduced a local $U(1)$ symmetry (See Eq.(1.4) given below), in addition to the detailed balance condition [33]. In order to have a healthy IR limit, however, they found that the latter has to be broken softly by adding all the low dimensional relevant terms. Even with these terms, the number of the independently coupling constants is reduced to 15.

the physical and geometrical interpretations of the gauge field A and Newtonian pre-potential φ . Specifically, after giving a brief introduction to the theory in Sec. II, we present all the spherically symmetric, stationary, vacuum solutions of the theory in closed forms in Sec. III. In Sec. IV, we consider the solar system tests by not taking A and φ as parts of the low energy 4-dimensional metric on which matter fields propagate, and find that theory is not consistent with observations as long as λ is not precisely equal to one, however small $|\lambda - 1|$ is. In Sec. V, we further study the limit $\lambda \rightarrow 1$ without assuming that the configuration is stationary. We find that, although the limit exists, it does not reduce to the Schwarzschild geometry. These results in Sec. IV and V strongly suggest that, in order for the theory to be consistent with the solar system tests, A and/or φ should enter the low-energy 4-dimensional metric. Thus, in Sec. VI, by requiring that the line element ds^2 be gauge-invariant not only under $\text{Diff}(M, \mathcal{F})$, but also under the $U(1)$ transformations, we propose that it should take the form,

$$ds^2 = -\mathcal{N}^2 c^2 dt^2 + g_{ij} (dx^i + \mathcal{N}^i dt) (dx^j + \mathcal{N}^j dt), \quad (1.5)$$

where

$$\begin{aligned} \mathcal{N} &\equiv N - \frac{v}{c^2} (A - \mathcal{A}), \quad \mathcal{N}^i \equiv N^i + N \nabla^i \varphi, \\ \mathcal{A} &\equiv -\dot{\varphi} + N^i \nabla_i \varphi + \frac{1}{2} N (\nabla_i \varphi)^2, \end{aligned} \quad (1.6)$$

where v is a dimensionless coupling constant to be constrained by experiments/observations, and subjected to radiative corrections. ∇_i denotes the covariant derivative with respect to the 3-metric g_{ij} . With such replacements, in this section we show explicitly that the resulted metrics are consistent with observations for both $\lambda = 1$ and $\lambda \neq 1$. With these replacements, one also sees clearly the physical and geometric meanings of A and φ . Our main results are summarized and discussed in Sec. VII.

Note that solar system tests were studied in other versions of the HL theory previously [43]. However, to our best knowledge, in the current paper it is the first time to consider the problem in the general covariant theory of the HL gravity with the projectability condition and an arbitrary coupling constant λ , while the case with $\lambda = 1$ was studied in [44].

In addition, all the high-order derivative terms of curvatures are negligible in the IR. Then, test particles move along geodesics, as shown explicitly in [45, 46] by using optical geometric approximations. Therefore, to have a consistent treatment, when we consider solar system tests, we ignore all the corrections from these high-order terms.

II. GENERAL COVARIANT HL THEORY

To realize the enlarged symmetry (1.4), HMT observed that the linearized (minimal) HL theory has a global $U(1)$

symmetry for $\lambda = 1$. This symmetry can be promoted to a local one by introducing a gauge field A , with which it was found that the scalar degree of freedom is eliminated [38]. When they lifted it to a full nonlinear theory, HMT found that the realization of the symmetry (1.4) requires introduction of an auxiliary scalar field φ , which was referred to as the “Newtonian prepotential.” Under the local $U(1)$, both A and φ transform as,

$$\delta_\alpha A = \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha, \quad (2.1)$$

while the lapse function N , the shift vector N^i and the 3-metric g_{ij} transform as,

$$\delta_\alpha N = 0, \quad \delta_\alpha N_i = N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0, \quad (2.2)$$

where α denotes the $U(1)$ generator, and $\dot{\alpha} \equiv \partial \alpha / \partial t$.

Under the coordinate transformations (1.2), φ transforms as a scalar, while A transforms as a vector under the time reparametrizations $t \rightarrow f(t')$, and as a scalar under the spatial transformations $\vec{x} \rightarrow \vec{\zeta}(t', \vec{x}')$, namely,

$$\begin{aligned} \delta A &= \zeta^i \partial_i A + \dot{f} A + f \dot{A}, \\ \delta \varphi &= f \dot{\varphi} + \zeta^i \partial_i \varphi. \end{aligned} \quad (2.3)$$

The metric components, N , N^i and g_{ij} , on the other hand, transform as

$$\begin{aligned} \delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\ \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}, \end{aligned} \quad (2.4)$$

under (1.2).

The HMT model was initially constructed in the case $\lambda = 1$, and it was soon found that it can be generalized to the case with an arbitrary λ [40], in which the spin-0 gravitons are also eliminated [40, 41], so the gravitational sector has the same degree of freedom as that in GR, i.e., only massless spin-2 gravitons exist.

For any given coupling constant λ , the total action can be written as [38–41],

$$S = \zeta^2 \int dt d^3 x N \sqrt{g} \left(\mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda + \zeta^{-2} \mathcal{L}_M \right), \quad (2.5)$$

where $g = \det g_{ij}$, and

$$\begin{aligned} \mathcal{L}_K &= K_{ij} K^{ij} - \lambda K^2, \\ \mathcal{L}_\varphi &= \varphi \mathcal{G}^{ij} \left(2K_{ij} + \nabla_i \nabla_j \varphi \right), \\ \mathcal{L}_A &= \frac{A}{N} \left(2\Lambda_g - R \right), \\ \mathcal{L}_\lambda &= (1 - \lambda) \left[(\nabla^2 \varphi)^2 + 2K \nabla^2 \varphi \right]. \end{aligned} \quad (2.6)$$

Here Λ_g is a coupling constant, and the Ricci and Riemann terms all refer to the three-metric g_{ij} , and

$$\begin{aligned} K_{ij} &= \frac{1}{2N} (-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i), \\ \mathcal{G}_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}. \end{aligned} \quad (2.7)$$

\mathcal{L}_M is the matter Lagrangian density, which in general is a function of all the dynamical variables, $U(1)$ gauge field, and the Newtonian prepotential, i.e., $\mathcal{L}_M = \mathcal{L}_M(N, N_i, g_{ij}, \varphi, A; \chi)$, where χ denotes collectively the matter fields. \mathcal{L}_V is an arbitrary $\text{Diff}(\Sigma)$ -invariant local scalar functional built out of the spatial metric, its Riemann tensor and spatial covariant derivatives, without the use of time derivatives.

Note the difference between the notations used here and the ones used in [38, 40]². In this paper, without further explanations, we shall use directly the notations and conventions defined in [47] and [39].

In [48], by assuming that the highest order derivatives are six, the minimum in order to have the theory to be power-counting renormalizable [4, 9], and that the theory preserves the parity, the most general form of \mathcal{L}_V was constructed and is given by,

$$\begin{aligned} \mathcal{L}_V = & \zeta^2 g_0 + g_1 R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij}) \\ & + \frac{1}{\zeta^4} (g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R_j^i R_k^j R_i^k) \\ & + \frac{1}{\zeta^4} [g_7 (\nabla R)^2 + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk})], \end{aligned} \quad (2.8)$$

where the coupling constants g_s ($s = 0, 1, 2, \dots, 8$) are all dimensionless, and

$$\Lambda = \frac{1}{2} \zeta^2 g_0, \quad (2.9)$$

is the cosmological constant. The relativistic limit in the IR requires

$$g_1 = -1, \quad \zeta^2 = \frac{1}{16\pi G}. \quad (2.10)$$

Then, the corresponding field equations are given in Appendix A.

III. SPHERICAL VACUUM SOLUTIONS

Spherically symmetric static vacuum spacetimes with projectability condition in the HMT setup were studied systematically in [44–46, 49]. In particular, the ADM quantities can be cast in the form [50, 51],

$$\begin{aligned} N = 1, \quad N^i \partial_i = e^{\mu-\nu} \partial_r, \\ g_{ij} dx^i dx^j = e^{2\nu} dr^2 + r^2 d^2\Omega, \end{aligned} \quad (3.1)$$

in the spherical coordinates $x^i = (r, \theta, \phi)$, where $d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2$, and

$$\mu = \mu(r), \quad \nu = \nu(r). \quad (3.2)$$

The corresponding timelike Killing vector is $\xi = \partial_t$. In the diagonal case, we have $\mu = -\infty$. With the gauge freedom of the local $U(1)$ symmetry, without loss of the generality, we can always fix the gauge by setting

$$\varphi = 0. \quad (3.3)$$

Then, we find that

$$F_\varphi^{ij} = 0, \quad F_\varphi^i = 0, \quad \mathcal{L}_\varphi = \mathcal{L}_\lambda = 0, \quad (3.4)$$

and

$$\begin{aligned} K_{ij} &= e^{\mu+\nu} (\mu' \delta_i^r \delta_j^r + r e^{-2\nu} \Omega_{ij}), \\ R_{ij} &= \frac{2\nu'}{r} \delta_i^r \delta_j^r + e^{-2\nu} [r\nu' - (1 - e^{2\nu})] \Omega_{ij}, \\ \pi_{ij} &= \frac{e^{\mu+\nu}}{r} [2\lambda + (\lambda - 1)r\mu'] \delta_i^r \delta_j^r \\ &\quad + r e^{\mu-\nu} [2\lambda - 1 + \lambda r\mu'] \Omega_{ij} \\ \mathcal{L}_K &= -\frac{e^{2(\mu-\nu)}}{r^2} [4\lambda - 2 + 4\lambda r\mu' + r^2(\lambda - 1)(\mu')^2], \\ \mathcal{L}_A &= \frac{2A}{r^2} [e^{-2\nu} (1 - 2r\nu') + (\Lambda_g r^2 - 1)], \end{aligned} \quad (3.5)$$

where $\Omega_{ij} \equiv \delta_i^\theta \delta_j^\theta + \sin^2\theta \delta_i^\phi \delta_j^\phi$ and $A = A(r)$. The expression for \mathcal{L}_V is very complicated and shall not be given explicitly here.

In the vacuum case, we have

$$J^t = J_A = J_\varphi = 0, \quad J_i = 0, \quad \tau_{ij} = 0. \quad (3.6)$$

Then, the Hamiltonian and momentum constraints (A.1) and (A.2) reduce, respectively, to

$$\int r^2 e^\nu (\mathcal{L}_K + \mathcal{L}_V) dr = 0, \quad (3.7)$$

$$a(r)h'' + b(r)h' + c(r)h = 0, \quad (3.8)$$

where

$$\begin{aligned} a(r) &= (1 - \lambda)r^2 f^2, \\ b(r) &= \frac{1}{2}(1 - \lambda)r f (4f - r f'), \\ c(r) &= -\frac{1}{2}(1 - \lambda) [r^2 (f f'' - f'^2) + 4f^2] - r f f', \end{aligned} \quad (3.9)$$

with

$$f(r) = e^{-2\nu}, \quad h(r) = e^{\mu-\nu}. \quad (3.10)$$

Equation (A.4) reads,

$$(1 - \lambda)(r h''' + 4h'') + d(r)h' + e(r)h = 0, \quad (3.11)$$

where

$$\begin{aligned} d(r) &= \frac{1}{4r f^2} \left\{ 4f^2 + 3(1 - \lambda)r^2 (f')^2 \right. \\ &\quad \left. + 4f[\Lambda_g r^2 - 1 - (1 - \lambda)r^2 f''] \right\}, \\ e(r) &= \frac{1}{4r f^3} \left\{ 3(\lambda - 1)r^2 (f')^3 + f f' [2(1 - r^2 \Lambda_g) \right. \\ &\quad \left. + (1 - \lambda)r(4f' + 5r f'')] + 2f^2 [(2\lambda - 1)f' \right. \\ &\quad \left. + 4r\Lambda_g - (1 - \lambda)r(2f'' + r f''')] \right\}. \end{aligned} \quad (3.12)$$

² In particular, we have $K_{ij} = -K_{ij}^{HMT}$, $\Lambda_g = \Omega^{HMT}$, $\varphi = -\nu^{HMT}$, $\mathcal{G}_{ij} = \Theta_{ij}^{HMT}$, where quantities with the super-indices “HMT” are those used in [38, 40].

Equation (A.5), on the other hand, yields,

$$(rf)' - (1 - \Lambda_g r^2) = 0, \quad (3.13)$$

while the dynamical equations (A.7) read

$$\left(\frac{A}{f^{1/2}}\right)' + \frac{G(r)}{2rf^{3/2}} = 0, \quad (3.14)$$

$$2r^2 f A'' + r(2f + rf') A' + r(f' + 2\Lambda_g r) A + H(r) = 0, \quad (3.15)$$

where $G(r)$ and $H(r)$ are defined in Eq.(B.1).

It should be noted that not all of the above equations are independent. In fact, Eq.(3.11) can be obtained from Eqs.(3.8) and (3.13), while Eq.(3.15) can be obtained from Eqs.(3.14), (3.8) and (3.13). Therefore, *in the present case there are only three independent differential equations, (3.8), (3.13), and Eqs.(3.14), for the three unknowns, (f, h, A)*³. In particular, from Eq.(3.13), we find that the general solution for f is given by

$$f(r) = 1 - \frac{2B}{r} - \frac{1}{3}\Lambda_g r^2, \quad (3.16)$$

where B is an integration constant.

Note that the momentum constraint (3.8) is a linear second-order ordinary differential equation for $h(r)$, and in principle one can integrate it to find $h(r)$ for the general solution $f(r)$ given above. Once $h(r)$ is found, one can integrate Eq.(3.14) to obtain A ,

$$A(r) = f^{1/2}(r) \left(A_0 - \frac{1}{2} \int \frac{G(r)dr}{r f^{3/2}(r)} \right), \quad (3.17)$$

where A_0 is an integration constant, and $G(r)$ is given by Eq.(B.1).

On the other hand, for the general solution (3.16), the potential \mathcal{L}_V defined by Eq.(2.8) is given by [45],

$$\mathcal{L}_V = 2\Lambda + \frac{1}{36r^9\zeta^4} (\alpha_0 + \alpha_1 r + \alpha_2 r^3 + \alpha_3 r^9), \quad (3.18)$$

where

$$\begin{aligned} \alpha_0 &= -216B^3(g_6 + 30g_8), \\ \alpha_1 &= 3240B^2g_8, \\ \alpha_2 &= 216B^2[(2g_5 + 2g_6 - 5g_8)\Lambda_g + g_3\zeta^2], \\ \alpha_3 &= 8\Lambda_g[4(9g_4 + 3g_5 + g_6)\Lambda_g^2 + 6\zeta^2(3g_2 + g_3)\Lambda_g \\ &\quad - 9\zeta^4]. \end{aligned} \quad (3.19)$$

All the solutions with $\lambda = 1$ were found in [44, 49], and their global structures were systematically studied in [45].

Therefore, in the rest of this section, we consider only the case where $\lambda \neq 1$. The case $\lambda \neq 1$ was also studied in [52], but only approximate solutions were found.

When $\lambda \neq 1$, a particular solution of Eq.(3.8) is $h(r) = 0$. Then, from Eqs.(B.1) and (3.16) we can see that $A(r)$ now is independent of λ , and the corresponding solutions will be the same as those given in the case $\lambda = 1, h = 0, f \neq 0$ [44, 49]. Therefore, in the following, we consider only the case where $h(r) \neq 0$. It is found convenient to consider the four cases, $\Lambda_g = 0 = B$; $\Lambda_g = 0, B \neq 0$; $\Lambda_g \neq 0, B = 0$; and $B\Lambda_g \neq 0$, separately.

$$1. \quad \Lambda_g = 0 = B$$

In this case, the momentum constraint (3.8) reduces to

$$r^2 h'' + 2rh' - 2h = 0, \quad (3.20)$$

which has the general solution,

$$h(r) = C_1 r + \frac{C_2}{r^2}, \quad (3.21)$$

where C_1 and C_2 are two integration constants. Inserting it into Eq.(3.17), we find that

$$A(r) = A_0 - \frac{3C_2^2}{8r^4} + \frac{1}{8} [3(1 - 3\lambda)C_1^2 + 2\Lambda] r^2. \quad (3.22)$$

$$2. \quad \Lambda_g = 0, B \neq 0$$

When $\Lambda_g = 0$ and $B \neq 0$, the momentum constraint (3.8) reduces to

$$\begin{aligned} &r(r - 2B)h'' + (2r - 5B)h' \\ &- \frac{2}{r} \left(\frac{r^2 - 5Br + 5B^2}{r - 2B} - B\varpi \right) h = 0, \end{aligned} \quad (3.23)$$

where $\varpi \equiv 1/(\lambda - 1)$. Note that when $\lambda = 1$, we must have $B = 0$, and Eq.(3.23) is identically satisfied for any $h(r)$, as noticed previously. When $\lambda \neq 1$, setting $x = r/(2B)$, $h(x) = h_0(r)h_1(x)$, Eq.(3.23) takes the form,

$$x(1-x)\frac{d^2 h_1}{dx^2} + p(x)\frac{dh_1}{dx} - q(x)h_1 = 0, \quad (3.24)$$

where

$$\begin{aligned} p(x) &= 2x(1-x)\frac{h_0'}{h_0} + \frac{5-4x}{2}, \\ q(x) &= x(x-1)\frac{h_0''}{h_0} + \frac{4x-5}{2}\frac{h_0'}{h_0} \\ &\quad + \frac{4x^2 - 10x + 5}{2x(1-x)} + \frac{\varpi}{x}. \end{aligned} \quad (3.25)$$

Assuming that Eq.(3.24) takes the form of the hypergeometric differential equation,

$$x(1-x)\frac{d^2 h_1}{dx^2} + [c - (a+b+1)x]\frac{dh_1}{dx} - abh_1 = 0, \quad (3.26)$$

³ Certainly, such obtained solutions must satisfy the global constraint (3.7).

where a , b and c are constants, from Eq.(3.25) we find

$$2x(1-x)\frac{h'_0}{h_0} + \frac{5-4x}{2} = c - (a+b+1)x, \quad (3.27)$$

$$x(x-1)\frac{h''_0}{h_0} + \frac{4x-5}{2}\frac{h'_0}{h_0} + \frac{4x^2-10x+5}{2x(1-x)} + \frac{\varpi}{x} = -ab. \quad (3.28)$$

Eq.(3.27) has the solution,

$$h_0(x) = (x-1)^{\frac{1}{4}(3+2a+2b-2c)} x^{\frac{1}{4}(2c-5)}, \quad (3.29)$$

for which Eq.(3.28) is satisfied identically, provided that

$$(a-b)^2 - 9 = 0, \quad (3.30)$$

$$2ab - c(a+b+1) + 2(5+\varpi) = 0, \quad (3.31)$$

$$4c^2 - 8c - 45 - 16\varpi = 0. \quad (3.32)$$

A solution of the above equations is given by

$$\begin{aligned} a &= \frac{1}{4}(7+\lambda_0), \quad b = \frac{1}{4}(\lambda_0-5), \\ c &= \frac{1}{2}(2+\lambda_0), \quad \lambda_0 = \sqrt{49+16\varpi}. \end{aligned} \quad (3.33)$$

Thus, the general solution of $h(r)$ takes the form,

$$\begin{aligned} h(r) &= h_0(r) \left[a_1 F(a, b; c; x) \right. \\ &\quad \left. + a_2 x^{-\frac{1}{2}\lambda_0} F\left(\frac{7-\lambda_0}{4}, \frac{-5-\lambda_0}{4}; \frac{2-\lambda_0}{2}; x\right) \right], \end{aligned} \quad (3.34)$$

where a_1 and a_2 are two integration (possibly complex) constants, and $F(a, b; c; z)$ is the hypergeometric function [53] with $F(a, b; c; 0) = 1$. Inserting Eq.(3.33) into Eq.(3.29), we find that

$$h_0(r) = \left(\frac{r}{2B} - 1\right)^{1/2} \left(\frac{r}{2B}\right)^{\frac{\lambda_0-3}{4}}. \quad (3.35)$$

On the other hand, for $\Lambda_g = 0$ Eq.(3.17) becomes,

$$A(r) = \sqrt{1 - \frac{2B}{r}} \left(A_0 - \int \frac{P(r)}{\sqrt{1 - \frac{2B}{r}}} dr \right), \quad (3.36)$$

where

$$\begin{aligned} P(r) &= \frac{1}{4(2B-r)^3} \left\{ [(9-25\lambda)B^2 + 2(1-2\lambda)r^2 \right. \\ &\quad + 4Br(5\lambda-2)]h^2 - 2r(2B-r)[(5\lambda-1)B \\ &\quad - 2r\lambda]hh' - (r-2B)^2 \left[\frac{4B}{r} + \frac{12B^3}{r^7\zeta^4}(22g_5 \right. \\ &\quad + 25g_6 - 20g_8) - \frac{2B^2}{r^6\zeta^4}(72g_5 + 81g_6 - 63g_8) \\ &\quad \left. \left. - g_3 \frac{2B^2}{r^4\zeta^2} - 2\Lambda r^2 - r^2(1-\lambda)(h')^2 \right] \right\}. \end{aligned} \quad (3.37)$$

The Hamiltonian constraint (3.7) now reads,

$$\int \frac{r^2 (\mathcal{L}_V + \mathcal{L}_K) dr}{\sqrt{1 - \frac{2B}{r}}} = 0, \quad (3.38)$$

where \mathcal{L}_V is given by Eq.(3.18) with $\Lambda_g = 0$, and

$$\begin{aligned} \mathcal{L}_K &= \left(\frac{2}{r^2} + \frac{B^2}{(r-2B)^2 r^2} \right) h^2 + \frac{2Bhh'}{2Br-r^2} \\ &\quad + (h')^2 - \lambda \left(h' + \frac{5B-2r}{r(2B-r)} h \right)^2. \end{aligned} \quad (3.39)$$

$$3. \quad B=0, \quad \Lambda_g \neq 0$$

When $B=0$, $\Lambda_g \neq 0$, the momentum constraint (3.8) reduces to

$$\begin{aligned} h'' + \frac{1}{r} \left(1 - \frac{3}{\Lambda_g r^2 - 3} \right) h' \\ - \frac{18(\lambda-1) - 3(5\lambda-7)\Lambda_g r^2 + (\lambda-3)\Lambda_g^2 r^4}{(\lambda-1)r^2(\Lambda_g r^2 - 3)^2} h = 0. \end{aligned} \quad (3.40)$$

Note that when $\lambda=1$ we have $\Lambda_g h = 0$. For $\lambda \neq 1$, Eq.(3.40) has the general solution,

$$\begin{aligned} h &= (1-z) \left\{ \frac{b_1}{r^2} F\left(-\frac{\lambda_1}{2}, \frac{\lambda_1}{2}; -\frac{1}{2}; z\right) \right. \\ &\quad \left. + b_2 r F\left(\frac{3-\lambda_1}{2}, \frac{3+\lambda_1}{2}; \frac{5}{2}; z\right) \right\}, \end{aligned} \quad (3.41)$$

where b_1 and b_2 are constants, and

$$z = \frac{1}{3}\Lambda_g r^2, \quad \lambda_1 = \sqrt{\frac{\lambda-3}{\lambda-1}}. \quad (3.42)$$

Then, Eq.(3.17) becomes,

$$A(r) = \sqrt{3 - \Lambda_g r^2} \left(A_0 - \int \frac{Q(r)}{\sqrt{3 - \Lambda_g r^2}} dr \right), \quad (3.43)$$

where

$$\begin{aligned} Q(r) &= \frac{1}{12(3 - \Lambda_g r^2)^3 r} \left\{ 9 \left[\lambda(\Lambda_g r^2 - 6)^2 - 3 \left[6 \right. \right. \right. \\ &\quad \left. \left. + (\Lambda_g r^2 - 4)\Lambda_g r^2 \right] \right] h^2 + 18(3 - \Lambda_g r^2)r \left[6\lambda \right. \\ &\quad \left. - (\lambda+1)\Lambda_g r^2 \right] hh' + (3 - \Lambda_g r^2)^2 r^2 \left[6\Lambda_g \right. \\ &\quad \left. - 18\Lambda + 9(\lambda-1)(h')^2 + \frac{4\Lambda_g^2}{\zeta^2}(3g_2 + g_3) \right. \\ &\quad \left. \left. + \frac{8\Lambda_g^3}{\zeta^4}(9g_4 + 3g_5 + g_6) \right] \right\}. \end{aligned} \quad (3.44)$$

The Hamiltonian constraint (3.7), on the other hand, takes the form,

$$\int \frac{r^2 (\mathcal{L}_V + \mathcal{L}_K) dr}{\sqrt{1 - \frac{1}{3}\Lambda_g r^2}} = 0, \quad (3.45)$$

where \mathcal{L}_K takes the same form of Eq.(3.39) but now with $h(r)$ given by Eq.(3.41) and \mathcal{L}_V given by,

$$\mathcal{L}_V = 2\Lambda + \frac{\alpha_3}{36\zeta^4}, \quad (3.46)$$

as can be seen from Eqs.(3.18) and (3.19).

$$4. \quad B \neq 0, \Lambda_g \neq 0$$

In this case, Eqs.(3.8) becomes,

$$\begin{aligned} & r(6B - 3r + \Lambda_g r^3) h'' + (15B - 6r + \Lambda_g r^3) h' \\ & - \left(\frac{6B - 2\Lambda_g r^3}{r(\lambda - 1)} + \frac{18(5B^2 - 5Br + r^2)}{r(6B - 3r + \Lambda_g r^3)} \right. \\ & \left. + \frac{3\Lambda_g(16B - 5r) + \Lambda_g^2 r^6}{r(6B - 3r + \Lambda_g r^3)} \right) h = 0. \end{aligned} \quad (3.47)$$

Setting

$$r = 2Bx, \quad \Lambda_0 = \frac{4}{3}B^2\Lambda_g, \quad (3.48)$$

and $h(r) = h_0(x)h_1(x)$, where

$$h_1(x) = \exp \int \frac{2c - 5 - 2x[a + b - 1 + (\Lambda_0 - e)x^2]}{4x(1 - x + \Lambda_0)} dx, \quad (3.49)$$

we find that Eq.(3.47) reduces to

$$\begin{aligned} & x(1 - x + \Lambda_0 x^3) h_0'' + [c - (a + b + 1)x + ex^3] h_0' \\ & - (ab + kx^2) h_0 = 0, \end{aligned} \quad (3.50)$$

but now with

$$\begin{aligned} a &= \frac{\lambda_0 + 7}{4}, \quad b = \frac{\lambda_0 - 5}{4}, \\ c &= \frac{\lambda_0 + 2}{2}, \quad e = \frac{\Lambda_0}{2}(\lambda_0 + 5), \\ k &= -\frac{3\Lambda_0}{8}(\lambda_0 + 7 + 8\varpi). \end{aligned} \quad (3.51)$$

When c is not an integral, expanding $h_0(x)$ in the form,

$$h_0 = A_1 \sum_{i=1}^{\infty} a_i x^i + A_2 \sum_{i=1}^{\infty} b_i x^{i+1-c}, \quad (3.52)$$

where A_1 and A_2 are two constants, we find that in terms of the two arbitrary constants a_0 and b_0 , the coefficients

a_i and b_i ($i \neq 0$) are given by,

$$\begin{aligned} a_1 &= \frac{ab}{c} a_0, \\ a_2 &= \frac{ab(a + b + ab + 1)}{2c(c + 1)} a_0, \\ a_3 &= \left[\frac{k}{3(c + 2)} + \frac{ab(a + b + ab + 1)(2a + 2b + ab + 4)}{6c(c + 1)(c + 2)} \right] a_0, \\ a_4 &= \left[\frac{(k - e)ab}{4c(c + 3)} + \frac{(a + b + ab + 1)(2a + 2b + ab + 4)}{24c(c + 1)(c + 2)(c + 3)} \right. \\ & \quad \left. \times (3a + 3b + ab + 9)ab + \frac{k(3a + 3b + ab + 9)}{12(c + 2)(c + 3)} \right] a_0, \\ b_1 &= \frac{ab + (1 - c)(a + b - c + 1)}{2 - c} b_0, \\ b_2 &= \frac{ab + (1 - c)(a + b - c + 1)}{2(2 - c)(3 - c)} \\ & \quad \times [ab + (2 - c)(a + b - c + 2)] b_0, \end{aligned} \quad (3.53)$$

and

$$\begin{aligned} a_j &= \frac{ab + (j - 1)(a + b + j - 1)}{j(j - 1 + c)} a_{j-1} \\ & \quad - \frac{(j - 3)(j\Lambda_0 - 4\Lambda_0 + e) - k}{j(j - 1 + c)} a_{j-3}, \quad j \geq 5 \\ b_j &= -\frac{(j - 2 - c)[(j - 3 - c)\Lambda_0 + e] - k}{j(j - 3 + c)} b_{j-3} \\ & \quad + \frac{ab + (j - c)(a + b + j - c)}{j(j - 3 + c)} b_{j-1}, \quad j \geq 3 \end{aligned} \quad (3.54)$$

Note that one can always set $A_1 = A_2 = 1$, by redefining the two arbitrary constants a_0 and b_0 .

On the other hand, when $c = 1 + m$ (where m is an integral), we can use the Frobenius method to solve Eq.(3.50). Let us first write $h_0(x)$ in the form,

$$h_0 = A_1 \sum_{i=0}^{\infty} a_i x^i + A_2 x^{-m} \left[\ln x \sum_{i=m}^{\infty} \bar{a}_i x^i + \sum_{i=0}^{\infty} b_i x^i \right], \quad (3.55)$$

where

$$\begin{aligned} \frac{c - (a + b + 1)x + ex^3}{1 - x + \Lambda_0 x^3} &= \sum_{i=0}^{\infty} c_i x^i, \\ \frac{-abx - kx^3}{1 - x + \Lambda_0 x^3} &= \sum_{i=0}^{\infty} d_i x^i, \end{aligned} \quad (3.56)$$

then, we can obtain the coefficients a_i , \bar{a}_i and b_i in terms of the two arbitrary constants c_0 and d_0 , which are given by

$$a_i = -\frac{\sum_{k=1}^i a_{i-k} [c_k(i - k) + d_k]}{i(i - 1 + c_0) + d_0},$$

$$\begin{aligned}
\bar{a}_i &= -\frac{\sum_{k=1}^i \bar{a}_{i-k} [c_k(i-k-m) + d_k]}{(i-m)(i-m-1+c_0) + d_0}, \\
b_i &= -\frac{\bar{a}_i(c_0-1-2m+2i)}{(i-m)(i-m-1+c_0) + d_0} \\
&\quad -\frac{\sum_{k=1}^i \{\bar{a}_{i-k}c_k + b_{i-k}[c_k(i-k-m) + d_k]\}}{(i-m)(i-m-1+c_0) + d_0}.
\end{aligned} \tag{3.57}$$

IV. FAILURE IN SOLAR SYSTEM TESTS

The solar system tests are usually written in terms of the Eddington parameters, by following the so-called “parameterized post-Newtonian” (PPN) approach, introduced initially by Eddington [54]. These parameters are often written in terms of the line element in its diagonal form,

$$ds^2 = -e^{2\Psi(r)} d\tau^2 + e^{2\Phi(r)} dr^2 + r^2 d\Omega^2. \tag{4.1}$$

Then, the gravitational field, produced by a point-like and motion-less particle with mass M , is given by

$$\begin{aligned}
e^{2\Psi} &= 1 - 2\left(\frac{GM}{c^2 r}\right) + 2(\beta - \gamma)\left(\frac{GM}{c^2 r}\right)^2 + \dots, \\
e^{2\Phi} &= 1 + 2\gamma\left(\frac{GM}{c^2 r}\right) + \dots,
\end{aligned} \tag{4.2}$$

where β and γ are the Eddington parameters. For the solar system, we have $r_g \equiv GM_\odot/c^2 \simeq 1.5$ km, and its radius is $r_\odot \simeq 1.392 \times 10^6$ km. So, within the solar system the dimensionless quantity $\chi[\equiv GM/(rc^2)]$ in most cases is much less than one, $\chi \leq r_g/r_\odot \leq 10^{-6}$. The Shapiro delay of the Cassini probe [55], and the solar system ephemerides [56] yield, respectively, the bounds [57],

$$\begin{aligned}
\gamma - 1 &= (2.1 \pm 2.3) \times 10^{-5}, \\
\beta - 1 &= (-4.1 \pm 7.8) \times 10^{-5}.
\end{aligned} \tag{4.3}$$

GR predicts $\beta = 1 = \gamma$ precisely. To study the solar system tests in the HL theory, we may first transform the above experimental results in terms of the ADM line element with the projectability condition,

$$ds^2 = -dt^2 + e^{2\Omega} (dr + e^{\Gamma-\Omega} dt)^2 + r^2 d\Omega^2, \tag{4.4}$$

for which it can be shown that [44],

$$\begin{aligned}
\Gamma &= \frac{1}{2} \ln \left\{ 2c^2 \left[\left(\frac{GM}{c^2 r} \right) - (\beta - \gamma) \left(\frac{GM}{c^2 r} \right)^2 + \dots \right] \right\}, \\
\Omega &= (\gamma - 1) \left(\frac{GM}{c^2 r} \right) + \dots
\end{aligned} \tag{4.5}$$

In the case $\lambda = 1$, two different identifications were prescribed. One was to consider A as part of the metric via the replacement [38],

$$N \rightarrow N - \frac{1}{c^2} A. \tag{4.6}$$

With such an identification, the diagonal solution [38, 44, 49],

$$\begin{aligned}
N &= 1, \quad N^i = 0, \quad f = 1 - \frac{2m}{r}, \\
A &= 1 - A_0 \sqrt{1 - \frac{2m}{r}}, \quad \varphi = 0, \quad (\lambda = 1),
\end{aligned} \tag{4.7}$$

produces exactly the Schwarzschild solution in the form (4.1) with $\Psi = -\Phi = \frac{1}{2} \ln(f)$. Note that in writing Eq.(4.7), the speed of light appearing in Eq.(4.6) had been set to one. As a result, the theory is consistent with observations [38].

However, the solution (4.7) is not unique, and there exists a larger class of non-diagonal solutions given by [44],

$$\begin{aligned}
\Gamma &= \ln h(r) \\
&= \frac{1}{2} \ln \left(\frac{2B}{r} + \frac{1}{3} \Lambda r^2 - 2A(r) + \frac{2}{r} \int^r A(r') dr' \right), \\
\Omega &= 0, \quad \varphi = 0, \quad (\lambda = 1),
\end{aligned} \tag{4.8}$$

where the gauge field $A(r)$ is undetermined. If one does not consider the gauge field A as a part of metric [44], but simply considers it as representing a degree of freedom of the gravitational field, as the Brans-Dicke scalar field in the Brans-Dicke theory [58], one finds that the above solutions are consistent with all the solar system tests, provided that [44],

$$A(r) = \mathcal{O} \left[\left(\frac{GM}{c^2 r} \right)^2 \right]. \tag{4.9}$$

In the rest of this section we shall follow the second prescription, i.e., setting directly,

$$(\Gamma, \Omega) = (\mu, \nu), \tag{4.10}$$

and verify whether this prescription can be generalized to the case $\lambda \neq 1$. As to be shown below, the answer is unfortunately negative.

To this goal, we first note that the cosmological constant Λ has negligible effects within the solar system. In addition, the spatial curvature of the solar system is negligible. In fact, for the metric (4.4), it takes the form,

$$\begin{aligned}
R &= \frac{2}{r^2} [1 - e^{-2\Omega} (1 - 2r\Omega')] \\
&\simeq \frac{8(\gamma - 1)^2}{r_g^2} \left(\frac{GM}{c^2 r} \right)^3,
\end{aligned} \tag{4.11}$$

for $r \gg r_g \equiv GM/c^2$. Note that in writing the last step of the above equation, we had used Eq.(4.5). Thus,

in the solar system we have $\Lambda_g = R/2 < 10^{-28} \text{ km}^{-2}$. Therefore, without loss of generality, we set

$$\Lambda = \Lambda_g = g_s = 0, \quad (s \geq 2). \quad (4.12)$$

Then, the solutions are those given by Eqs.(3.16) and (3.34), from which we find that

$$\begin{aligned} \nu &= -\frac{1}{2} \ln \left(1 - \frac{2B}{r} \right) \\ &\simeq \epsilon \left(\frac{GM}{c^2 r} \right) + \epsilon^2 \left(\frac{GM}{c^2 r} \right)^2 \\ &\quad + \mathcal{O} \left[\left(\frac{GM}{c^2 r} \right)^3 \right], \end{aligned} \quad (4.13)$$

where $\epsilon \equiv Bc^2/(GM)$. Comparing the above with Eq.(4.5), we find that

$$\gamma - 1 = \epsilon = \frac{Bc^2}{GM} \leq 10^{-4}, \quad (4.14)$$

for $M = M_\odot$. As a result, we have

$$x = \frac{r}{2B} = \frac{1}{2(\gamma - 1)} \left(\frac{GM}{c^2 r} \right)^{-1} \gg 1. \quad (4.15)$$

From the relations,

$$\begin{aligned} F(a, b; d; x) &= (1 - x)^{-b} F \left(b, d - a; d; \frac{x}{x - 1} \right), \\ F(a, b; d; 1) &= \frac{\Gamma(d)\Gamma(d - a - b)}{\Gamma(d - a)\Gamma(d - b)}, \end{aligned} \quad (4.16)$$

where the last expression holds only for $d \neq 0, -1, -2, \dots$, $\text{Re}(d - a - b) > 0$, we find from Eq.(3.34) that $h(r)$ has the asymptotical form,

$$h(r) \simeq D_1 r, \quad (x \gg 1), \quad (4.17)$$

with

$$\begin{aligned} D_1 &= \frac{1}{2B} \left[a_1 (-1)^b F(b, c - a; c; 1) \right. \\ &\quad \left. + a_2 (-1)^{\hat{b}} F(\hat{b}, \hat{c} - \hat{a}; \hat{c}; 1) \right], \\ \hat{a} &= \frac{7 - \lambda_0}{4}, \quad \hat{b} = -\frac{5 + \lambda_0}{4}, \quad \hat{c} = \frac{2 - \lambda_0}{2}. \end{aligned} \quad (4.18)$$

Note that Eq.(4.17) can be also obtained directly from Eq.(3.26), which reads

$$x^2 h_1'' + (a + b + 1)x h_1' + a b h_1 = 0, \quad (4.19)$$

for $x \gg 1$. Eq.(4.19) has the general solution,

$$h_1(x) = d_1 x^{(5 - \lambda_0)/4} + d_2 x^{-(7 + \lambda_0)/4}, \quad (4.20)$$

where d_1 and d_2 are two integration constants. On the other hand, from Eq.(3.29) we find that

$$h_0(x) \simeq x^{(\lambda_0 - 1)/4}. \quad (4.21)$$

Then, we obtain

$$h(x) = h_0(x)h_1(x) \simeq d_1 x + \frac{d_2}{x^2} \simeq d_1 x, \quad (4.22)$$

which is precisely the solution given by Eq.(4.17) with $D_1 = d_1/(2B)$. Hence, we obtain

$$\begin{aligned} \Gamma(r) &= \frac{1}{2} \ln \left(\frac{h^2}{f} \right) \\ &= \frac{1}{2} \ln \left\{ \left(\frac{c^2 r}{GM} \right)^2 \left[1 + \left(\frac{2c^2 B}{GM} \right) \left(\frac{GM}{c^2 r} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{2c^2 B}{GM} \right)^2 \left(\frac{GM}{c^2 r} \right)^2 + \mathcal{O}(\chi^3) \right] \right\} \\ &\quad + \ln \left(\frac{D_1 GM}{c^2} \right). \end{aligned} \quad (4.23)$$

This is quite different from that given by Eq.(4.5) with any choice of a_1 , a_2 , B , as long as λ is not exactly equal to one. Therefore, the static vacuum solutions given by Eqs.(3.16) and (3.34) with the condition (4.12) is inconsistent with the solar system tests, when the prescription (4.10) is used.

V. MOST GENERAL ANSATZ WITH SPHERICAL SYMMETRY

In the previous section, based on the stationary ansatz (3.1), we have seen that the prescription (4.10) leads to failure in solar system tests for $\lambda \neq 1$, however small $|\lambda - 1|$ is. Hence, in the next section we shall consider another prescription. Before that, however, in this section let us consider the most general ansatz with spherical symmetry and show that the prescription (4.10) never recovers the Schwarzschild geometry in the $\lambda \rightarrow 1$ limit with $\Lambda_g = 0$. This confirms that a prescription beyond (4.10) is absolutely necessary.

In order to find the most general ansatz, note that one can always choose time and spatial coordinates so that $N = 1$ and $N^i = 0$ at least locally. One can also set $\varphi = 0$ by the $U(1)$ gauge freedom. With

$$N = 1, \quad N^i = 0, \quad \varphi = 0, \quad (5.1)$$

it is obvious that the most general ansatz with spherical symmetry is ⁴,

$$g_{ij} dx^i dx^j = e^{2B(t,x)} dx^2 + e^{2C(t,x)} d\Omega^2, \quad A = A(t, x). \quad (5.2)$$

Independent equations are the equation of motion for the gauge field A , the x -component of the momentum constraint and the xx -component of the dynamical equation.

⁴ One must not confuse with the function $B(t, x)$ used in this section and the constant B used in the expression of $f(r)$ in the previous and next sections.

The equation of motion for the gauge field A is written as

$$\partial_x \left[e^{-2B+3C} (\partial_x C)^2 + \frac{1}{3} \Lambda_g e^{3C} - e^C \right] = 0, \quad (5.3)$$

leading to the general solution

$$B = \frac{3}{2}C + \frac{1}{2} \ln \left[\frac{(\partial_x C)^2}{F(t) + e^C - (\Lambda_g/3)e^{3C}} \right], \quad (5.4)$$

where $F(t)$ is an arbitrary function of time. The momentum constraint is

$$\begin{aligned} & \partial_x \partial_t C + \partial_x C \partial_t C - \partial_x C \partial_t B \\ & + (\lambda - 1) \left[\partial_x \partial_t C + \frac{1}{2} \partial_x \partial_t B \right] = 0. \end{aligned} \quad (5.5)$$

By using the solution (5.4), this equation is reduced to an equation for C :

$$\begin{aligned} & \partial_t [\Lambda_g e^{2C} - 3F e^{-C}] \\ & = (\lambda - 1) \left\{ c_1 \left[\frac{\partial_x^2 \partial_t C}{(\partial_x C)^2} - \frac{(\partial_x \partial_t C)(\partial_x^2 C)}{(\partial_x C)^3} \right] \right. \\ & \quad \left. + c_2 \frac{\partial_x \partial_t C}{\partial_x C} + c_3 \partial_t C + c_4 \right\}, \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} c_1 &= \frac{1}{\Delta} \left[3F^2 e^{-C} + 2(3 - \Lambda_g e^{2C})F \right. \\ & \quad \left. + \frac{1}{3} e^{-C} (\Lambda_g e^{3C} - 3e^C)^2 \right], \\ c_2 &= \frac{1}{6\Delta} \left[63F^2 e^{-C} + 3(39 - 11\Lambda_g e^{2C})F \right. \\ & \quad \left. + 2(2\Lambda_g^2 e^{5C} - 15\Lambda_g e^{3C} + 27e^C) \right], \\ c_3 &= \frac{1}{2\Delta} \left[3(3\Lambda_g e^{2C} - 1)F + 4\Lambda_g e^{3C} \right], \\ c_4 &= \frac{3}{2\Delta} (1 - \Lambda_g e^{2C}) \partial_t F, \\ \Delta &= F + e^C - (\Lambda_g/3)e^{3C}. \end{aligned} \quad (5.7)$$

Finally, the dynamical equation can be considered as an equation determining the gauge field A . With the prescription (4.10) where A does not participate in the geometry nor in solar system tests, the dynamical equation is not of our interest.

Now let us expand C by $(\lambda - 1)$ as

$$C(t, x) = \sum_{n=0}^{\infty} C_n(t, x) (\lambda - 1)^n. \quad (5.8)$$

We shall see below that the momentum constraint equation (5.6) can be solved iteratively order by order in the $(\lambda - 1)$ expansion, under the condition (5.12) below.

First, the zeroth order solution C_0 is obtained as a solution to the following algebraic equation

$$\Lambda_g e^{2C_0(t, x)} - 3F(t) e^{-C_0(t, x)} = G(x), \quad (5.9)$$

where $G(x)$ is an arbitrary function of x . Next, let us show by induction that the n -th order solution $C_n(t, x)$ can be obtained by solving (5.6) order by order. For this purpose, let us expand the expression inside the squared bracket on the left hand side of (5.6) by $(\lambda - 1)$ as

$$\Lambda_g e^{2C} - 3F(t) e^{-C} = \sum_{n=0}^{\infty} \mathcal{G}_n(t, x) (\lambda - 1)^n, \quad (5.10)$$

according to the expansion (5.8). It is easy to understand that \mathcal{G}_n has the form

$$\mathcal{G}_n = (2\Lambda_g e^{2C_0} + 3F(t) e^{-C_0}) C_n + \tilde{\mathcal{G}}_n, \quad (5.11)$$

where $\tilde{\mathcal{G}}_n$ depends only on C_i ($i = 1, 2, \dots, n-1$). Thus, provided that

$$\begin{aligned} & 2\Lambda_g e^{2C_0} + 3F(t) e^{-C_0} \neq 0, \\ & F + e^{C_0} - (\Lambda_g/3) e^{3C_0} \neq 0, \\ & \partial_x C_0 \neq 0, \end{aligned} \quad (5.12)$$

we obtain

$$\begin{aligned} C_n(t, x) &= -\frac{1}{\Delta_1} \left[\tilde{\mathcal{G}}_n[C_1, \dots, C_{n-1}; t, x] \right. \\ & \quad \left. - \int_{t_0}^t dt' S_n[C_1, \dots, C_{n-1}; t', x] \right], \\ \Delta_1 &= 2\Lambda_g e^{2C_0} + 3F(t) e^{-C_0}, \end{aligned} \quad (5.13)$$

where S_n is the n -th order part of the right hand side of (5.6), which also depends only on C_i ($i = 1, 2, \dots, n-1$), and t_0 is an initial time. Note that the initial value of C_n at $t = t_0$ has been set to zero by redefinition of $G(x)$ and that the change due to the shift of the initial time t_0 corresponds to redefinition of $G(x)$. From this result, it is obvious by induction that the solution of the form (5.8) can be obtained up to any order of the expansion.

Let us consider the zero-th order solution (5.9) with $\Lambda_g = 0$. In order for the expansion w.r.t. $(\lambda - 1)$ to make sense, the condition (5.12) must be satisfied. In particular, $F(t)$ should be non-vanishing. Otherwise, the denominator on the r.h.s. of (5.13) would vanish. We thus assume that $F(t) \neq 0$.

We would like to see if the Schwarzschild geometry is recovered in the limit $\lambda \rightarrow 1$ with $\Lambda_g = 0$ or not. One of the simplest ways is to calculate the 4-dimensional Einstein tensor for the 4-dimensional metric

$$ds_4^2 = -dt^2 + e^{2B_0(t, x)} dx^2 + e^{2C_0(t, x)} d\Omega^2, \quad (5.14)$$

where

$$B_0 = \frac{1}{2} \ln \left[\frac{27F(t)^2 (\partial_x G(x))^2}{G(x)^4 (3 - G(x))} \right], \quad C_0 = \ln \left[\frac{-3F(t)}{G(x)} \right]. \quad (5.15)$$

Non-vanishing components of the 4-dimensional Einstein tensor are

$$\begin{aligned} G_t^{(4)t} &= -\frac{3(\partial_t F)^2}{F^2}, \\ G_x^{(4)x} &= -\frac{1}{F^2} \left[(\partial_t F)^2 + 2F\partial_t^2 F + \frac{G^3}{27} \right], \\ G_\theta^{(4)\theta} &= -\frac{1}{F^2} \left[(\partial_t F)^2 + 2F\partial_t^2 F - \frac{G^3}{54} \right]. \end{aligned} \quad (5.16)$$

In order to recover the Schwarzschild metric, all of these components must vanish, leading to $\partial_t F = G = 0$. However, in this case the regularity of C_0 implies that $F = 0$, contradicting with the assumption $F \neq 0$. Note that $F \neq 0$ is a necessary condition for the continuity of the $\lambda \rightarrow 1$ limit.

If we set $F = 0$ then the $\lambda \rightarrow 1$ limit is singular. Thus, the zero-th order solution is not obtained as the $\lambda \rightarrow 1$ limit of a solution with $\lambda \neq 1$. Instead, it represents a solution with exactly $\lambda = 1$. In this case, (5.9) with $\Lambda_g = 0$ implies that $G = 0$ and leaves C_0 unspecified. There is a choice of C_0 giving rise to the Schwarzschild metric.

Just for completeness, let us consider the case with $F = 0$ and $\Lambda_g \neq 0$. In this case the $\lambda \rightarrow 1$ limit is continuous. However, the zeroth order solution is

$$B_0 = \frac{1}{2} \ln \frac{3(\partial_x G)^2}{4\Lambda_g G(G-3)}, \quad C_0 = \frac{1}{2} \ln \frac{G}{\Lambda_g}, \quad (5.17)$$

resulting in

$$G_t^{(4)t} = -\Lambda_g, \quad G_x^{(4)x} = G_\theta^{(4)\theta} = -\frac{1}{3}\Lambda_g. \quad (5.18)$$

Hence, the Schwarzschild metric is not recovered in this case unless the limit $\Lambda_g \rightarrow 0$ is taken. If we take this limit then the $\lambda \rightarrow 1$ limit becomes singular. Thus, again, the Schwarzschild solution is not obtained as the $\lambda \rightarrow 1$ limit of a solution with $\lambda \neq 1$. Instead, it represents a solution with exactly $\lambda = 1$.

In summary, if $\lambda = 1$ and $\Lambda_g = 0$ exactly then the Schwarzschild metric is one of solutions. However, if we consider $\lambda \neq 1$ and $\Lambda_g = 0$, then the Schwarzschild metric is never recovered in the limit $\lambda \rightarrow 1$. This conclusion is based on the prescription (4.10) and thus implies that a prescription beyond (4.10) is absolutely necessary.

VI. PHYSICAL INTERPRETATION OF A AND φ

In this section, we shall show that a proper generalization of the prescription of (4.6) can lead to solutions that are consistent with solar system tests even for $\lambda \neq 1$. From such a generalization, the physical and geometrical interpretations of the gauge field A and Newtonian prepotential φ also become clear.

A. General Coupling of A and φ to Metric

To the above claim, we first note that under the $U(1)$ transformations, the ADM quantities transform as [38, 41],

$$\begin{aligned} \delta_\alpha N &= 0, \quad \delta_\alpha N_i = N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0, \\ \delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = -\alpha, \end{aligned} \quad (6.1)$$

where $\delta_\alpha F = \tilde{F} - F$, $\alpha = \alpha(t, x)$ is the generator of the local $U(1)$ gauge symmetry. From the above we find that

$$\begin{aligned} \delta_\alpha \mathcal{A} &= \dot{\alpha} - N^i \nabla_i \alpha, \\ \delta_\alpha \sigma &= 0, \quad \delta_\alpha \mathcal{N}^i = 0, \end{aligned} \quad (6.2)$$

where \mathcal{A} is defined in Eq.(1.6), and

$$\sigma \equiv A - \mathcal{A}, \quad \mathcal{N}^i \equiv N^i + N \nabla^i \varphi. \quad (6.3)$$

If we require that the line element ds^2 be gauge-invariant not only under $\text{Diff}(M, \mathcal{F})$ (1.2), but also under the enlarged symmetry (1.4), then ds^2 defined by,

$$ds^2 \equiv -\mathcal{N}^2 c^2 dt^2 + g_{ij} (dx^i + \mathcal{N}^i dt) (dx^j + \mathcal{N}^j dt), \quad (6.4)$$

has the desired properties, where

$$\mathcal{N} \equiv N - \frac{v}{c^2} (A - \mathcal{A}), \quad (6.5)$$

where v is a dimensionless coupling constant subjected possibly to radiative corrections. Similar to N , such defined \mathcal{N} is also dimensionless, $[\mathcal{N}] = 0$. With this prescription, one can see that the Newtonian prepotential φ is tightly related to the shift vector \mathcal{N}^i , while the geometrical lapse function \mathcal{N} is related to both A and φ . In addition, since

$$\begin{aligned} [dx] &= -1, \quad [dt] = -z, \quad [c] = \frac{[dx]}{[dt]} = z - 1, \\ [N] &= 0, \quad [N^i] = z - 1, \quad [g_{ij}] = 0, \\ [A] &= [\mathcal{A}] = 2(z - 1), \quad [\varphi] = z - 2, \\ [\alpha] &= z - 2, \end{aligned} \quad (6.6)$$

we find that

$$[ds] = -1, \quad (6.7)$$

i.e., it has the dimension of length. Moreover, with the gauge choice $\varphi = 0$ and setting $v = 1$, Eq.(6.5) reduces to Eq.(4.6).

In the Newtonian limit, we have [38, 59]

$$g_{00} = -\left(1 + \frac{2\phi}{c^2} + \mathcal{O}(\epsilon)\right), \quad g_{0i} = \mathcal{O}(\epsilon), \quad (6.8)$$

in the coordinates $x^\mu = (ct, x^i)$, where $\epsilon \equiv |v/c| \ll 1$, and v denotes the typical velocity of the system concerned.

Comparing it with the metric given by Eqs.(6.3)-(6.5), we find that the Newtonian potential ϕ is given by

$$\phi = -v(A + \dot{\varphi}) - \frac{1}{2}N^i N_i + (v-1) \left(N^i + \frac{1}{2}\nabla^i \varphi \right) \nabla_i \varphi, \quad (6.9)$$

with

$$N = 1, \quad \frac{1}{c} |2N_i + \nabla_i \varphi| = \mathcal{O}(\epsilon). \quad (6.10)$$

To study further the meaning of the above prescription and the physical interpretations of the gauge field A and Newtonian prepotential φ , let us turn to the solar system tests again.

B. Solutions with the Gauge $A = 0$

For the ADM decomposition (3.1) without fixing the U(1) gauge, there are three independent equations, given by Eqs.(D.1) - (D.3). To solve these equations, let us first note that the prescriptions of Eqs.(6.3) - (6.5) do not change the spatial metric g_{ij} . As a result, the constraint on the spatial curvature R takes the same form of Eq.(4.11). Therefore, in the present case the condition (4.12) can be still imposed safely. In particular, with the gauge $A = 0$ [cf. Appendix C for different gauge choices.], Eqs.(D.1) and (D.2) for $\lambda = 1$ have the solutions,

$$f = 1 - \frac{2B}{r}, \quad h = -f\varphi', \quad (\lambda = 1, A = 0), \quad (6.11)$$

where φ must satisfy the dynamical equation Eqs.(D.3), which now reads,

$$\left(1 - \frac{2B}{r}\right)^2 [(\varphi')^2]' + \left(1 - \frac{2B}{r}\right) \frac{B}{r^2} (\varphi')^2 + \frac{2B}{r^2} = 0. \quad (6.12)$$

The general solutions are given by,

$$\varphi(r) = \varphi_0 \pm \int \left(\frac{2r}{r-2B} + \varphi_1 \sqrt{\frac{r}{r-2B}} \right)^{1/2} dr, \quad (6.13)$$

where φ_0 and φ_1 are integrations constants. Substituting the above into Eq.(6.3) we find that $\mathcal{N}^i = 0$. Then, Eq.(6.4) reduces to,

$$ds^2 = -\mathcal{N}^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d^2\Omega, \quad (6.14)$$

where

$$\mathcal{N}^2 = \frac{1}{4} \left[2 - v \left(2 - \frac{\varphi_1(\epsilon\chi - 1)}{\sqrt{1 - \epsilon\chi}} \right) \right]^2, \quad f = 1 - \frac{2B}{r}, \quad (\lambda = 1, A = 0). \quad (6.15)$$

When $\chi \ll 1$, we have

$$\mathcal{N}^2 = \Upsilon^2 \left(1 - \frac{C_1 v \epsilon}{2\Upsilon} \chi - \frac{C_1(v-1)v\epsilon^2}{8\Upsilon^2} \chi^2 + \mathcal{O}(\chi^3) \right), \quad \frac{1}{f} = 1 + \epsilon\chi + \epsilon^2\chi^2 + \mathcal{O}(\chi^3), \quad (6.16)$$

where $\Upsilon \equiv v(1 + \varphi_1/2) - 1$. The factor Υ appearing in the expression of \mathcal{N} can be dropped by rescaling $t \rightarrow \Upsilon t$. Then, comparing Eq.(6.16) with Eq.(4.2) we find that

$$B = \gamma \left(\frac{GM}{c^2} \right), \quad \beta = \frac{1}{2}(\gamma + 1), \quad v = \frac{2}{2 + (\gamma - 1)\varphi_1}. \quad (6.17)$$

For $\varphi_1 \simeq \mathcal{O}(1)$, we obtain the constraint $|v - 1| < \mathcal{O}(10^{-5})$ from (4.3). For extremely large value of φ_1 , say $\varphi_1 \simeq \mathcal{O}(10^5)$, $|v - 1| \simeq \mathcal{O}(1)$ is also allowed. However, we consider this large value of φ_1 unrealistic and consider the case with $\varphi_1 = \mathcal{O}(1)$ only. Note that the Schwarzschild solution corresponds to $B = GM/c^2$, $v = 1$.

C. Solutions with the Gauge $\varphi \cdot A \neq 0$

On the other hand, in the case $\lambda \neq 1$ let us consider the gauge $h = 0$. Then, Eqs.(D.1) and (D.2) yield,

$$f(r) = 1 - \frac{2B}{r}, \quad \varphi(r) = \int r^{\frac{1-\lambda_0}{4}} \left[b_1 F\left(\frac{9-\lambda_0}{4}; \frac{-3-\lambda_0}{4}; \frac{2-\lambda_0}{2}; x\right) + b_2 r^{\frac{\lambda_0}{2}} F\left(\frac{3+\lambda_0}{4}; \frac{9+\lambda_0}{4}; \frac{2+\lambda_0}{2}; x\right) \right] dr + \varphi_0, \quad (6.18)$$

where $b_{1,2}$ are constants, and λ_0 is given by Eq.(3.33). Substituting the above into Eq.(D.3), we find that

$$A(r) = \sqrt{1 - \frac{2B}{r}} \left(A_0 - \int \frac{\hat{P}(r)}{\sqrt{1 - \frac{2B}{r}}} dr \right), \quad (6.19)$$

with

$$\hat{P}(r) = \frac{1}{4(2B-r)r^2} \left\{ \left[(21-33\lambda)B^2 + 2(3-4\lambda)r^2 + 2Br(16\lambda-11) \right] (\varphi')^2 - 4Br - r^2(r-2B)^2(\lambda-1)(\varphi'')^2 \right\} + \frac{\lambda-1}{2r} \varphi' [(2B-r)r\varphi''' - 2B\varphi'']. \quad (6.20)$$

When $b_1 = b_2 = 0$, the above solutions reduce to

$$\varphi(r) = \varphi_0, \quad A = 1 - A_0 \sqrt{1 - \frac{2B}{r}}. \quad (6.21)$$

Substituting it into the metric (6.4), and considering the gauge choice $h = 0$, we find that it takes exactly the form of Eq.(6.14) with the metric coefficients given by Eq.(6.15) and with the replacement $\varphi_1 \rightarrow -2A_0$. Thus, the PPN parameters β and γ are given by (6.17) with φ_1 replaced by $-2A_0$. For $A_0 \simeq \mathcal{O}(1)$, we obtain the constraint $|v - 1| < \mathcal{O}(10^{-5})$ again from (4.3). Therefore, the prescription (6.4) leads to consistent results with solar system tests even for $\lambda \neq 1$.

VII. CONCLUSIONS

In this paper, we have studied spherically symmetric, stationary vacuum configurations in the general covariant theory of the Hořava-Lifshitz gravity with the projectability condition $N = N(t)$, and an arbitrary value of the coupling constant λ [38–41]. In particular, in Sec. III we have obtained all the solutions with the assumed symmetry in closed forms.

When applying these solutions to the solar system tests (Sec. IV), we have shown explicitly that the ADM-type identification (4.10) between the metric coefficients and the basic quantities N, N^i and g_{ij} do not render the $\lambda \neq 1$ solutions consistent with solar system tests, no matter how small $|\lambda - 1|$ is. (On the other hand, when $\lambda = 1$ exactly, there is a spherically-symmetric, stationary vacuum solution which is consistent with the solar system tests [44].)

To show that this is indeed the case in more general situations, we have devoted Sec. V to consider the most general ansatz (5.1) and (5.2) with spherical symmetry and shown that the prescription (4.10) never recovers the Schwarzschild geometry in the $\lambda \rightarrow 1$ limit with $\Lambda_g = 0$. Thus, one needs either to invent a mechanism to restrict λ precisely to its relativistic value $\lambda_{GR} = 1$, or to consider the gauge field A and/or the Newtonian prepotential φ as parts of the 4-dimensional metric on which matter fields propagate.

In the case $\lambda = 1$, HMT proposed the identification (4.6) [38] but clearly it is not gauge-invariant. Requiring the line element be gauge-invariant not only under $\text{Diff}(M, \mathcal{F})$ (1.2), but also under the enlarged symmetry (1.4), in Sec. VI we have proposed the identification (6.4), where v is a dimensionless constant to be constrained by observations/experiments. When $v = 1$, it reduces to (4.6) in the gauge $\varphi = 0$. Applying such a prescription to the cases both with $\lambda = 1$ and with $\lambda \neq 1$, we have shown that the resulted metric is indeed consistent with the solar system tests, provided that $|v - 1| < 10^{-5}$. With such identifications, one can also see the physical and geometrical roles that A and φ play. In particular, the Newtonian prepotential φ is tightly related to the

shift vector, while the geometrical lapse function \mathcal{N} is related to both A and φ .

Finally, we note that it still remains to be understood how to obtain the prescription (6.4) (with $v \simeq 1$) from the action principle⁵. Actually, in the UV, N and $A - \mathcal{A}$ have different scaling dimensions and thus, it is not easy to imagine how their linear combination can universally enter the UV action of matter fields. On the other hand, in the IR, N and $A - \mathcal{A}$ have the same scaling dimensions (they are actually dimensionless) and thus, the prescription (6.4) is not forbidden a priori. It is therefore important to investigate whether the prescription (6.4) can emerge in the IR and, if it does, how.

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Appendix A: Field Equations

Corresponding to the actions (2.5), the Hamiltonian and momentum constraints are given respectively by,

$$\begin{aligned} \int d^3x \sqrt{g} \left[\mathcal{L}_K + \mathcal{L}_V - \varphi \mathcal{G}^{ij} \nabla_i \nabla_j \varphi - (1 - \lambda) (\nabla^2 \varphi)^2 \right] \\ = 8\pi G \int d^3x \sqrt{g} J^t, \end{aligned} \quad (\text{A.1})$$

$$\nabla^j \left[\pi_{ij} - \varphi \mathcal{G}_{ij} - (1 - \lambda) g_{ij} \nabla^2 \varphi \right] = 8\pi G J_i, \quad (\text{A.2})$$

where

$$\begin{aligned} J^t &\equiv 2 \frac{\delta(N\mathcal{L}_M)}{\delta N}, \\ \pi_{ij} &\equiv -K_{ij} + \lambda K g_{ij}, \\ J_i &\equiv -N \frac{\delta \mathcal{L}_M}{\delta N^i}. \end{aligned} \quad (\text{A.3})$$

Variation of the action (2.5) with respect to φ and A yield, respectively,

$$\begin{aligned} \mathcal{G}^{ij} (K_{ij} + \nabla_i \nabla_j \varphi) + (1 - \lambda) \nabla^2 (K + \nabla^2 \varphi) \\ = 8\pi G J_\varphi, \end{aligned} \quad (\text{A.4})$$

$$R - 2\Lambda_g = 8\pi G J_A, \quad (\text{A.5})$$

⁵ In [60] the coupling of the HL covariant theory with matter was considered from the action principle. It was shown that Newtonian gravity cannot be recovered in the weak gravitational field approximation, based on several assumptions, including the one that the coupling among matter, the gauge field A and the Newtonian prepotential φ be described by the recipe provided in [40].

where

$$J_\varphi \equiv -\frac{\delta \mathcal{L}_M}{\delta \varphi}, \quad J_A \equiv 2\frac{\delta(N\mathcal{L}_M)}{\delta A}. \quad (\text{A.6})$$

On the other hand, the dynamical equations now read,

$$\begin{aligned} & \frac{1}{N\sqrt{g}} \left\{ \sqrt{g} [\pi^{ij} - \varphi \mathcal{G}^{ij} - (1-\lambda)g^{ij}\nabla^2\varphi] \right\}_{,t} \\ &= -2(K^2)^{ij} + 2\lambda K K^{ij} \\ &+ \frac{1}{N} \nabla_k [N^k \pi^{ij} - 2\pi^{k(i} N^{j)}] \\ &- 2(1-\lambda) [(K + \nabla^2\varphi) \nabla^i \nabla^j \varphi + K^{ij} \nabla^2\varphi] \\ &+ (1-\lambda) [2\nabla^{(i} F_\varphi^{j)} - g^{ij} \nabla_k F_\varphi^k] \\ &+ \frac{1}{2} (\mathcal{L}_K + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda) g^{ij} \\ &+ F^{ij} + F_\varphi^{ij} + F_A^{ij} + 8\pi G \tau^{ij}, \end{aligned} \quad (\text{A.7})$$

where $(K^2)^{ij} \equiv K^{il} K_l^j$, $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$, and

$$\begin{aligned} F^{ij} &\equiv \frac{1}{\sqrt{g}} \frac{\delta(-\sqrt{g}\mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^8 g_s \zeta^{n_s} (F_s)^{ij}, \\ F_\varphi^{ij} &= \sum_{n=1}^3 F_{(\varphi,n)}^{ij}, \\ F_\varphi^i &= (K + \nabla^2\varphi) \nabla^i \varphi + \frac{N^i}{N} \nabla^2\varphi, \\ F_A^{ij} &= \frac{1}{N} [A R^{ij} - (\nabla^i \nabla^j - g^{ij} \nabla^2) A], \end{aligned} \quad (\text{A.8})$$

with $n_s = (2, 0, -2, -2, -4, -4, -4, -4, -4)$. The stress 3-tensor τ^{ij} is defined as

$$\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g_{ij}}, \quad (\text{A.9})$$

and the geometric 3-tensors $(F_s)_{ij}$ and $F_{(\varphi,n)}^{ij}$ are given in [42].

The matter components $(J^t, J^i, J_\varphi, J_A, \tau^{ij})$ satisfy the conservation laws,

$$\begin{aligned} & \int d^3x \sqrt{g} \left[\dot{g}_{kl} \tau^{kl} - \frac{1}{\sqrt{g}} (\sqrt{g} J^t)_{,t} + \frac{2N_k}{N\sqrt{g}} (\sqrt{g} J^k)_{,t} \right. \\ & \quad \left. - 2\dot{\varphi} J_\varphi - \frac{A}{N\sqrt{g}} (\sqrt{g} J_A)_{,t} \right] = 0, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} & \nabla^k \tau_{ik} - \frac{1}{N\sqrt{g}} (\sqrt{g} J_i)_{,t} - \frac{J^k}{N} (\nabla_k N_i - \nabla_i N_k) \\ & - \frac{N_i}{N} \nabla_k J^k + J_\varphi \nabla_i \varphi - \frac{J_A}{2N} \nabla_i A = 0. \end{aligned} \quad (\text{A.11})$$

Appendix B: G and H defined in Eqs.(3.14) and (3.15)

The functions G and H defined in Eqs.(3.14) and (3.15) are given by

$$\begin{aligned} G(r) &= rh[2h' - r(\lambda - 1)h''] + \frac{1}{2}r^2(\lambda - 1)h'^2 \\ &+ \frac{r}{2f} [r(\lambda - 1)f'' - 2(\lambda + 1)f'] h^2 \\ &- \frac{3r^2 f'^2}{8f^2} (\lambda - 1)h^2 + (4\lambda - 3)h^2 \\ &- \frac{1}{4\zeta^4 r^4} \left\{ 8(46g_4 + 17g_5 + 7g_6 + 28g_7 + 9g_8)f^3 \right. \\ &- 4[8g_7 f^{(4)} r^4 + 3g_8 f^{(4)} r^4 + 14g_2 \zeta^2 r^2 + 5g_3 \zeta^2 r^2 \\ &+ (48g_4 + 14g_5 + 3g_6 - 48g_7 - 16g_8)f'' r^2 \\ &- 2(48g_4 + 14g_5 + 3g_6 - 48g_7 - 16g_8)f' r + 180g_4 \\ &+ 66g_5 + 27g_6 + 48g_7 + 12g_8] f^2 + [-3(8g_7 \\ &+ 3g_8)(f'')^2 r^4 + (48g_4 + 22g_5 + 12g_6 + 48g_7 \\ &+ 13g_8)(f')^2 r^2 + 4(8g_2 r^2 \zeta^2 + 3g_3 r^2 \zeta^2 + 48g_4 \\ &+ 14g_5 + 3g_6 - 16g_7 - 4g_8)f'' r^2 - 2f'[3(32g_4 \\ &+ 12g_5 + 5g_6 - g_8)f'' r^2 + 2((8g_7 + 3g_8)f''' r^3 \\ &+ 96g_4 + 28g_5 + 6g_6 - 32g_7 - 8g_8)]r + 4[r^4 \zeta^4 \\ &+ 12g_2 r^2 \zeta^2 + 4g_3 r^2 \zeta^2 + 84g_4 + 30g_5 + 12g_6 - 8g_7 \\ &- 6g_8]] f + (32g_4 + 12g_5 + 5g_6 - g_8)r^3 (f')^3 \\ &+ 4[\zeta^4 \Lambda r^6 - \zeta^4 r^4 + 2g_2 \zeta^2 r^2 + g_3 \zeta^2 r^2 + 4g_4 \\ &+ 2g_5 + g_6] + r^2 (f')^2 [-8g_2 \zeta^2 r^2 - 3g_3 \zeta^2 r^2 \\ &+ (8g_7 + 3g_8)f'' r^2 - 48g_4 - 14g_5 \\ &- 3g_6 + 16g_7 + 4g_8] \left. \right\}, \\ H(r) &= 2\lambda r^2 h h'' + (\lambda + 1)r^2 (h')^2 + 4(2\lambda - 1) r h h' \\ &- \frac{(2\lambda + 1)r^2 f'}{f} h h' - \frac{r}{f} [\lambda r f'' + (2\lambda - 1)f'] h^2 \\ &+ \frac{r^2}{4f^2} (5\lambda + 1)(f')^2 h^2 \\ &+ \frac{1}{4r^4 \zeta^4} \left\{ 32(46g_4 + 17g_5 + 7g_6 + 28g_7 + 9g_8)f^3 \right. \\ &+ 4[8g_7 f^{(5)} r^5 + 3g_8 f^{(5)} r^5 + 48g_4 f''' r^3 \\ &+ 14g_5 f''' r^3 + 3g_6 f''' r^3 - 48g_7 f''' r^3 \\ &- 16g_8 f''' r^3 - 28g_2 \zeta^2 r^2 - 10g_3 \zeta^2 r^2 - 4(48g_4 \\ &+ 14g_5 + 3g_6 - 48g_7 - 16g_8)f'' r^2 + 6(2g_4 - 3g_5 \\ &- 4g_6 - 76g_7 - 25g_8)f' r - 720g_4 - 264g_5 \\ &- 108g_6 - 192g_7 - 48g_8] f^2 + 2[-16g_2 \zeta^2 f''' r^5 \\ &- 6g_3 \zeta^2 f''' r^5 + 3(32g_4 + 12g_5 + 5g_6 - g_8)(f'')^2 r^4 \\ &- 96g_4 f''' r^3 - 28g_5 f''' r^3 - 6g_6 f''' r^3 + 32g_7 f''' r^3 \end{aligned}$$

$$\begin{aligned}
& +8g_8 f''' r^3 + 48g_2 \zeta^2 r^2 + 16g_3 \zeta^2 r^2 - 3(112g_4 \\
& + 30g_5 + 4g_6 - 144g_7 - 47g_8)(f')^2 r^2 + f''(5(8g_7 \\
& + 3g_8)f''' r^3 + 8(48g_4 + 14g_5 + 3g_6 - 16g_7 \\
& - 4g_8))r^2 + f' \left(3(32g_4 + 12g_5 + 5g_6 - g_8)f''' r^3 \right. \\
& + (48g_4 - 2g_5 - 15g_6 - 240g_7 - 74g_8)f'' r^2 \\
& + 2(3(8g_7 + 3g_8)f^{(4)} r^4 + 2(14g_2 r^2 \zeta^2 + 5g_3 r^2 \zeta^2 \\
& + 36g_4 + 24g_5 + 18g_6 + 96g_7 + 24g_8)) \left. \right) r + 672g_4 \\
& + 240g_5 + 96g_6 - 64g_7 - 48g_8] f - (16g_4 + 10g_5 \\
& + 7g_6 + 48g_7 + 14g_8)r^3 (f')^3 + 8(-\zeta^4 \Lambda r^6 + 2g_2 \zeta^2 r^2 \\
& + g_3 \zeta^2 r^2 + 8g_4 + 4g_5 + 2g_6) - r f' [- (8g_7 \\
& + 3g_8)(f'')^2 r^4 + 2(8g_2 r^2 \zeta^2 + 3g_3 r^2 \zeta^2 + 48g_4 \\
& + 14g_5 + 3g_6 - 16g_7 - 4g_8)f'' r^2 + 4(r^4 \zeta^4 \\
& + 12g_2 r^2 \zeta^2 + 4g_3 r^2 \zeta^2 + 84g_4 + 30g_5 + 12g_6 \\
& - 8g_7 - 6g_8)] + 3r^2 (f')^2 [(8g_7 + 3g_8)f''' r^3 \\
& + (32g_4 + 12g_5 + 5g_6 - g_8)f'' r^2 + 96g_4 + 28g_5 \\
& + 6g_6 - 32g_7 - 8g_8] \left. \right\}, \tag{B.1}
\end{aligned}$$

where $f^{(n)} \equiv d^n f / dr^n$.

Appendix C: The U(1) Gauge Transformations and Gauge Choices

Under the U(1) gauge transformations (6.1), in the spherically symmetric case, the variables $(N, N^i, g_{ij}, A, \varphi)$ transform as,

$$\begin{aligned}
\delta_\alpha N &= 0, \quad \delta_\alpha N^i = \delta_r^i f \alpha', \quad \delta_\alpha g_{ij} = 0, \\
\delta_\alpha A &= \dot{\alpha} - h \alpha', \quad \delta_\alpha \varphi = -\alpha, \tag{C.1}
\end{aligned}$$

where $\alpha = \alpha(t, r)$. From these expressions, one can see that various gauges can be chosen.

$$\text{A.} \quad \varphi = 0$$

In this gauge, we have

$$\alpha = \varphi(t, r), \tag{C.2}$$

which is unique, and is the gauge used in Section III.

$$\text{B.} \quad A = 0$$

In this gauge, we have

$$\dot{\alpha} - h \alpha' = -A. \tag{C.3}$$

When $h = 0$, we have

$$\alpha(t, r) = - \int^t A(t', r) dt' + \alpha_0(r), \tag{C.4}$$

where $\alpha_0(r)$ is an arbitrary function of its indicated argument. Thus, in this case the gauge is fixed only up to an arbitrary function of r .

When $h \neq 0$, we introduce two new variables u and v via the relations,

$$\begin{aligned}
dt &= G dv + F du, \\
dr &= h(G dv - F du), \tag{C.5}
\end{aligned}$$

where F and G are functions of u and v only, and satisfy the integrability conditions,

$$F_{,v} - G_{,u} = 0, \tag{C.6}$$

$$(Fh)_{,v} + (Gh)_{,u} = 0. \tag{C.7}$$

Note that one should not consider Eq.(C.5) as coordinate transformations, because they are forbidden by $\text{Diff}(M, \mathcal{F})$, but rather a technique to solve Eq.(C.3). Then, in terms of u and v , Eq.(C.3) takes the form, $\alpha_{,u} = -FA$, which has the solution,

$$\alpha(t, r) = - \int^u F(u', v) A(u', v) du' + \alpha_1(v), \tag{C.8}$$

where α_1 is an arbitrary function of v only, and $u = u(t, r)$ and $v = v(t, r)$, given through Eqs.(C.5)-(C.7). Therefore, in the present case the gauge is fixed up to an arbitrary function of v .

$$\text{C.} \quad h = 0$$

In this gauge, we have

$$\alpha' = -\frac{h}{f}, \tag{C.9}$$

which has the solution,

$$\alpha(t, r) = - \int^r \frac{h(t, r') dr'}{f(t, r')} + \alpha_2(t), \tag{C.10}$$

where $\alpha_2(t)$ is an arbitrary function of t only.

Appendix D: Field Equations without Specifying the U(1) Gauge

It can be shown that in the spherically symmetric case, there are only three independent field equations: the constraint obtained from the variation of the gauge field A given by Eq.(A.4), the momentum constraint (A.2), and the rr-component of the dynamical equations (A.7). For

the ADM decomposition given by Eq.(3.1), they read, respectively,

$$(rf)' - (1 - \Lambda_g r^2) = 0, \quad (\text{D.1})$$

$$\begin{aligned} (1 - \lambda) \left\{ r^2 f^2 h'' - \frac{rf}{2} (rf' - 4f) h' - \left[2f^2 \right. \right. \\ \left. \left. - \frac{r^2}{2} (f')^2 + \frac{r^2}{2} f f'' \right] h - \frac{f^2}{2} \left[4f\varphi' - 4rf' \right. \right. \\ \left. \left. - r^2 f'' - (4rf + 3r^2 f')\varphi'' - 2rf\varphi''' \right] \right\} \\ - rf f' h + f^2 (f - 1 + \Lambda_g r^2) \varphi' = 0, \quad (\text{D.2}) \\ \frac{16}{r} A' + \frac{8}{r^2 f} (f - 1 + r^2 \Lambda_g) A \\ + \frac{4}{r^2} (3f - 1 + r^2 \Lambda_g) \varphi' (\varphi' + h) + \frac{16hh'}{rf} \\ + \frac{8h^2}{r^2 f^2} (f - 2rf') - \frac{2}{r^6 \zeta^4 f} \left[8(46g_4 + 17g_5 + 7g_6 \right. \\ \left. - 28g_7 + 9g_8) f^3 - 4 \left(- (8g_7 - 3g_8) f^{(4)} r^4 + (14g_2 \right. \right. \\ \left. \left. + 5g_3) \zeta^2 r^2 + (48g_4 + 14g_5 + 3g_6 + 48g_7 \right. \right. \\ \left. \left. - 16g_8) (rf'' - 2f') r + 180g_4 + 66g_5 + 27g_6 \right. \right. \\ \left. \left. + 12(g_8 - 4g_7) \right) f^2 + \left(4(r^4 \zeta^4 + 4(3g_2 + g_3) r^2 \zeta^2 \right. \right. \\ \left. \left. + 84g_4 + 30g_5 + 12g_6 + 8g_7 - 6g_8) + r((48g_4 \right. \right. \end{aligned}$$

$$\begin{aligned} + 22g_5 + 12g_6 - 48g_7 + 13g_8) r (f')^2 - 2(2(3g_8 \\ - 8g_7) f''' r^3 + 3(32g_4 + 12g_5 + 5g_6 - g_8) f'' r^2 \\ + 4(48g_4 + 14g_5 + 3g_6 + 16g_7 - 4g_8)) f' \\ + rf'' (3(8g_7 - 3g_8) f'' r^2 + 4((8g_2 + 3g_3) r^2 \zeta^2 + 48g_4 \\ + 14g_5 + 3g_6 + 16g_7 - 4g_8))) \Big) f + 4 \left(r^2 (r^2 (r^2 \Lambda - 1) \zeta^2 \right. \\ \left. + 2g_2 + g_3) \zeta^2 + 4g_4 + 2g_5 + g_6 \right) - r^2 (f')^2 ((8g_2 \\ + 3g_3) r^2 \zeta^2 + 48g_4 + 14g_5 + 3g_6 + 16g_7 - 4g_8 \\ + r((-32g_4 - 12g_5 - 5g_6 + g_8) f' + (8g_7 - 3g_8) r f'')) \Big) \\ + (1 - \lambda) \left\{ \left[4f'' - \frac{(f')^2}{f} + \frac{8f'}{r} - \frac{32f}{r^2} \right] \varphi^2 + 8f' \varphi' \varphi'' \right. \\ \left. + \frac{h^2}{r^2 f^3} \left[3r^2 (f')^2 - 32f^2 - 4rf(rf'' - 2f') \right] \right. \\ \left. + \frac{8}{f} h h'' + \left(\frac{16f'}{rf} - \frac{64}{r^2} + \frac{6(f')^2}{f^2} \right) \varphi' h \right. \\ \left. + \frac{8f'}{f} (2\varphi'' h - \varphi' h') + 8\varphi' h'' - \frac{4}{f} (h' + f\varphi'')^2 \right. \\ \left. + 8(h + f\varphi') \varphi''' \right\} = 0. \quad (\text{D.3}) \end{aligned}$$

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